Ασκηση

- Δείξτε ότι

$$\textbf{(a)}\quad x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, d\omega$$

$$\textbf{(b)}\quad X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$

Λύση

(a) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$

Substituting $t = 0$ in the preceding equation, we get

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,d\omega$$

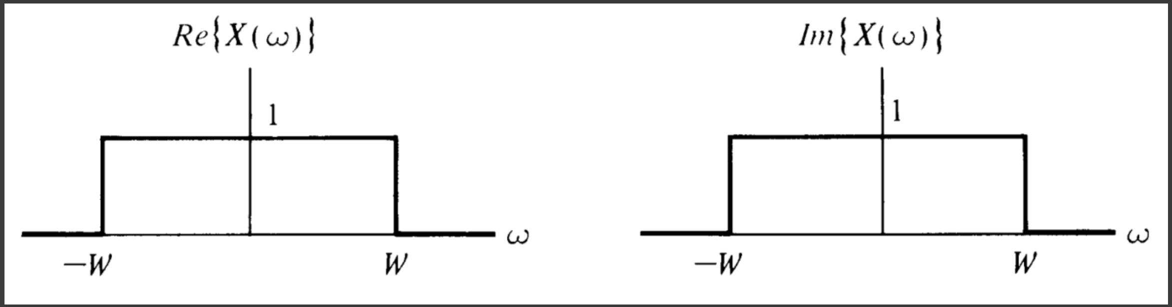
(b) $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$

Substituting $\omega = 0$ in the preceding equation, we get

$$X(0) = \int_{-\infty}^{\infty} x(t)\,dt$$

Ασκηση

- Έστω ότι για το x(t) ισχύει:



- Σχεδιάστε το πλάτος και τη φάση
- Είναι πραγματικό το x(t);

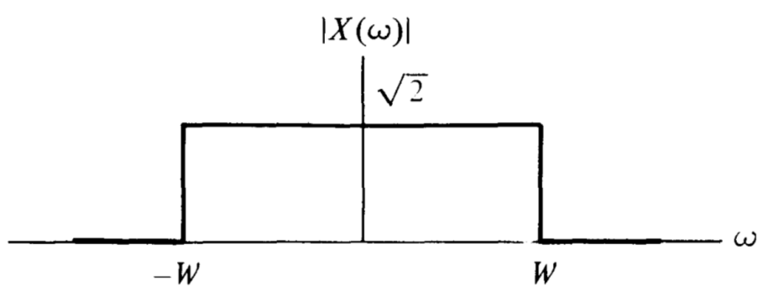
Λύση

The magnitude of $X(\omega)$ is given by

$$|X(\omega)| = \sqrt{X_R^2(\omega) + X_I^2(\omega)},$$

where $X_R(\omega)$ is the real part of $X(\omega)$ and $X_I(\omega)$ is the imaginary part of $X(\omega)$. It follows that

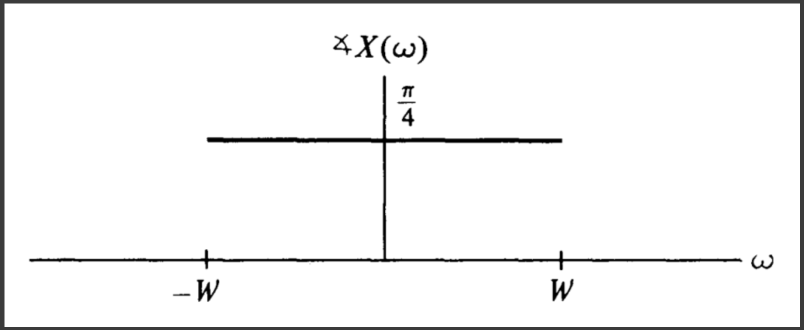
$$|X(\omega)| = \begin{cases} \sqrt{2}, & |\omega| < W, \\ 0, & |\omega| > W \end{cases}$$



Λύση

The phase of $X(\omega)$ is given by

$$\sphericalangle X(\omega) = \tan^{-1}\left(\frac{X_I(\omega)}{X_R(\omega)}\right) = \tan^{-1}(1), \qquad |\omega| < W$$



Λύση

- Το x(t) για να είναι πραγματικό θα πρέπει: $X(-\omega) = X^*(\omega)$

$$X(\omega) = \begin{cases} 1 + j, & |\omega| < W \\ 0, & \text{otherwise} \end{cases}$$

$$X(-\omega) = \begin{cases} 1 + j, & |\omega| < W \\ 0, & \text{otherwise} \end{cases}$$

$$X^*(\omega) = \begin{cases} 1 - j, & |\omega| < W \\ 0, & \text{otherwise} \end{cases}$$

- Δεν είναι πραγματικο!

- Από τη θεωρία (πινακας 2.1) γνωρίζουμε ότι το σήμα :

$$e^{-at}u(t),\ \mathrm{Re}(a) > 0$$

Έχει Fourier

$$\frac{1}{j\Omega + a}$$

- Υπολογίστε το Fourier του $e^{-a|t|}$

Λύση

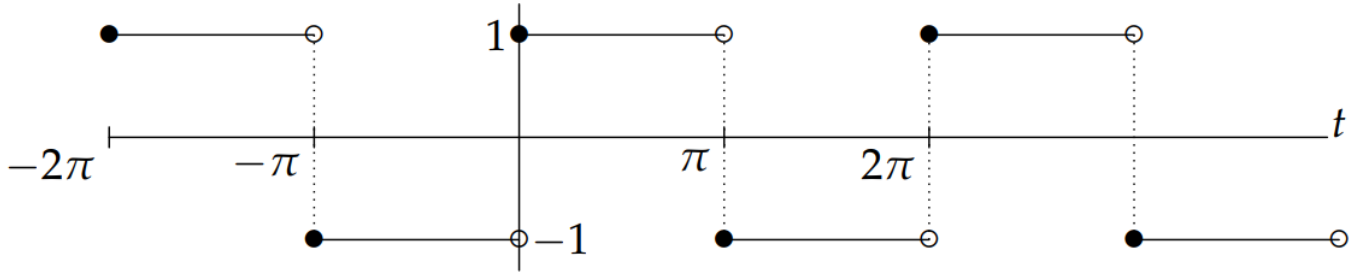
$$\mathcal{F}\{e^{-\alpha|t|}\} = \mathcal{F}\{e^{-\alpha t}u(t) + e^{\alpha t}u(-t)\}$$
$$= \frac{1}{\alpha + j\omega} + \frac{1}{\alpha - j\omega}$$
$$= \frac{2\alpha}{\alpha^2 + \omega^2}$$

Ασκηση

Example 1. Compute the Fourier series of $f(t)$, where $f(t)$ is the *square wave* with period 2π. which is defined over one period by

$$f(t) = \begin{cases} -1 & \text{for } -\pi \leq t < 0 \\ 1 & \text{for } 0 \leq t < \pi \end{cases} .$$

The graph over several periods is shown below.



Λυση

Solution. Computing a Fourier series means computing its Fourier coefficients. We do this using the integral formulas for the coefficients given with Fourier's theorem in the previous note. For convenience we repeat the theorem here.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)),$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\cos nt\,dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\sin nt\,dt$$

In applying these formulas to the given square wave function, we have to split the integrals into two pieces corresponding to where $f(t)$ is $+1$ and where it is -1. We find

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \int_{-\pi}^{0} (-1)\cdot\cos(nt)\,dt + \int_{0}^{\pi} (1)\cdot\cos(nt)\,dt.$$

Thus, for $n \neq 0$:

$$a_n = -\left.\frac{\sin(nt)}{n\pi}\right|_{-\pi}^{0} + \left.\frac{\sin(nt)}{n\pi}\right|_{0}^{\pi} = 0$$

and for $n = 0$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = 0.$$

Λυση

Likewise

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} -\sin(nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi}\sin(nt)\,dt$$

$$= \left.\frac{\cos(nt)}{n\pi}\right|_{-\pi}^{0} - \left.\frac{\cos(nt)}{n\pi}\right|_{0}^{\pi} = \frac{1-\cos(-n\pi)}{n\pi} - \frac{\cos(n\pi)-1}{n\pi}$$

$$= \frac{2}{n\pi}(1-\cos(n\pi)) = \frac{2}{n\pi}(1-(-1)^n) = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}.$$

We have used the simplification $\cos n\pi = (-1)^n$ to get a nice formula for the coefficients b_n. (Note: when you get $\cos n\pi$ in these calculations it's *always useful* to make this substitution.)

This then gives the Fourier series for $f(t)$:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(nt) = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right).$$

Προβλημα

Determine the Fourier transform of $x(t) = e^{-t/2}u(t)$

- $|X(\omega)|$
- $Re(X(\omega))$
- $Im(X(\omega))$

Λυση

The Fourier transform of $x(t)$ is

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} e^{-t/2}u(t)e^{-j\omega t}\,dt \qquad \text{(S9.1-1)}$$

Since $u(t) = 0$ for $t < 0$, eq. (S9.1-1) can be rewritten as

$$X(\omega) = \int_{0}^{\infty} e^{-(1/2+j\omega)t}\,dt = \frac{+2}{1 + j2\omega}$$

It is convenient to write $X(\omega)$ in terms of its real and imaginary parts:

$$X(\omega) = \frac{2}{1 + j2\omega}\left(\frac{1 - j2\omega}{1 - j2\omega}\right) = \frac{2 - j4\omega}{1 + 4\omega^2} = \frac{2}{1 + 4\omega^2} - j\,\frac{4\omega}{1 + 4\omega^2}$$

$$\text{Magnitude of } X(\omega) = \frac{2}{\sqrt{1 + 4\omega^2}}$$

Προβλημα

Compute the Fourier transform of each of the following signals:

(a) $[e^{-\alpha t} \cos \omega_0 t]u(t), \qquad \alpha > 0$

(b) $e^{-3|t|} \sin 2t$

Λυση

(a) $x(t) = e^{-\alpha t}\cos\omega_0 t u(t), \qquad \alpha > 0$
$= e^{-\alpha t}u(t)\cos(\omega_0 t)$

Therefore,

$$X(\omega) = \frac{1}{2\pi}\frac{1}{\alpha + j\omega} * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$
$$= \frac{1/2}{\alpha + j(\omega - \omega_0)} + \frac{1/2}{\alpha + j(\omega + \omega_0)}$$

(b) $x(t) = e^{-3|t|}\sin 2t$

$$e^{-3|t|} \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{6}{9 + \omega^2}$$

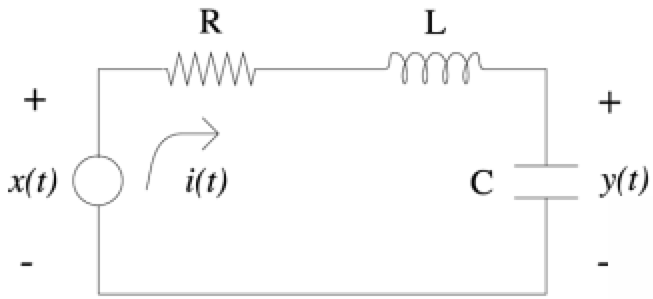
$$\sin 2t \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)], \qquad \omega_0 = 2$$

Therefore,

$$X(\omega) = \frac{1}{2\pi}\left(\frac{6}{9 + \omega^2}\right) * \left\{\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]\right\}$$
$$= \frac{j3}{9 + (\omega + 2)^2} - \frac{j3}{9 + (\omega - 2)^2}$$

Παραδείγματα Συστημάτων

Ex. #1 RLC circuit



$$R\ i(t) + L\frac{di(t)}{dt} + y(t) = x(t)$$

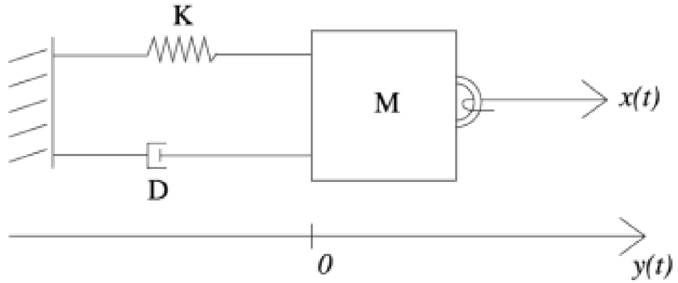
$$i(t) = C\frac{dy(t)}{dt}$$

$$\Downarrow$$

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

Παραδείγματα Συστημάτων

Ex. #2 Mechanical system



x(t) - applied force
K - spring constant
D - damping constant
y(t) - displacement from rest

Force Balance:

$$\mathrm{M}\frac{d^2y(t)}{dt^2} = x(t) - \mathrm{K}y(t) - \mathrm{D}\frac{dy(t)}{dt}$$

$$\Downarrow$$

$$\mathrm{M}\frac{d^2y(t)}{dt^2} + \mathrm{D}\frac{dy(t)}{dt} + \mathrm{K}y(t) = x(t)$$

Observation: Very different physical systems may be modeled mathematically in very similar ways.