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MODELING UNSTRUCTURED DECISION PROBLEMS – THE THEORY OF ANALYTICAL HIERARCHIES

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Quantitative modeling of unstructured decision problems with social implications is new and challenging and has pressing needs. A new approach to scaling using largest eigenvalues and reciprocal matrices and the effect of inconsistent judgment are introduced and relevant theory discussed. In this approach inconsistency is accepted as a fact but measured to determine how bad it is. Since most decision problems are hierarchical in form as they fulfill higher and still higher objectives, the appropriate structure for representation is a hierarchy. A new formal definition of a hierarchy is given and the notion of measurement with eigenvalues is extended to hierarchies. Both the eigenvalue approach to measurement and the hierarchical approach are illustrated with examples. Finally, unstructured problems are illustrated through applications of forward-backward planning, a two-point boundary value problem.

1. Introduction

Decision making is a practice we all engage in all the time. All human undertakings require initiative and action to choose one of several alternatives. Thus decision theory rather than being a theoretical concern must be a practical pursuit which shapes our thinking about choices and is influenced by our intuition and judgment as to how we like to make our choices. Some decision theory has been normative without regard to human preferences. Here we turn the subject around and look at it as a natural human preoccupation, formalize it and advocate its usage through participation.

Since the behavioral sciences have no laws or invariants one can usually produce a counter example to every seemingly good hypothesis. For any modeling to be useful in this field it cannot be of the universal kind one is accustomed to in the natural sciences. It must attempt to describe or solve today's problems *with* the people who have these problems. Thus, because of the lack of invariants, one can only describe the present happenings by cooperation with the people themselves and using their own judgments and interpretations guiding the model towards a satisfactory answer. Therefore useful modeling must be interactive and must include subjectivity arising out of the concerned party's experience rather than dictated by the modeler, who may reflect complete ignorance of the occurrences.

Decision theory is concerned with making an optimal choice among alternative outcomes. In order that the choice be rational, a way of making tradeoffs among the alternatives according to their various attributes must be known.

Decision problems can be represented as in fig. 1. The structured and semi-structured parts of the figure are rather well understood. Most complex real-life problems are unstructured and the task is to estimate both the possible outcomes and their corresponding probabilities. In order to decrease the guesswork in estimating alternative outcomes, a set of extreme outcomes or scenarios is first identified and a most likely one is computed as a weighted combination of these outcomes. Hierarchical analysis is a technique used to estimate the weights of the possible outcomes. This forward planning process is sharpened by adjoining to its policies those estimated to be effective in attaining known desired outcomes from the backward process.

Our first problem then is to identify alternative (forward) outcomes, together with the actors who by pursuing their objectives and policies (independently or through cooperation) attempt to bring about the likely outcomes. As a result of this interaction we ask "What is the outcome likely to emerge and what kind of measurement do we use to estimate this outcome?"

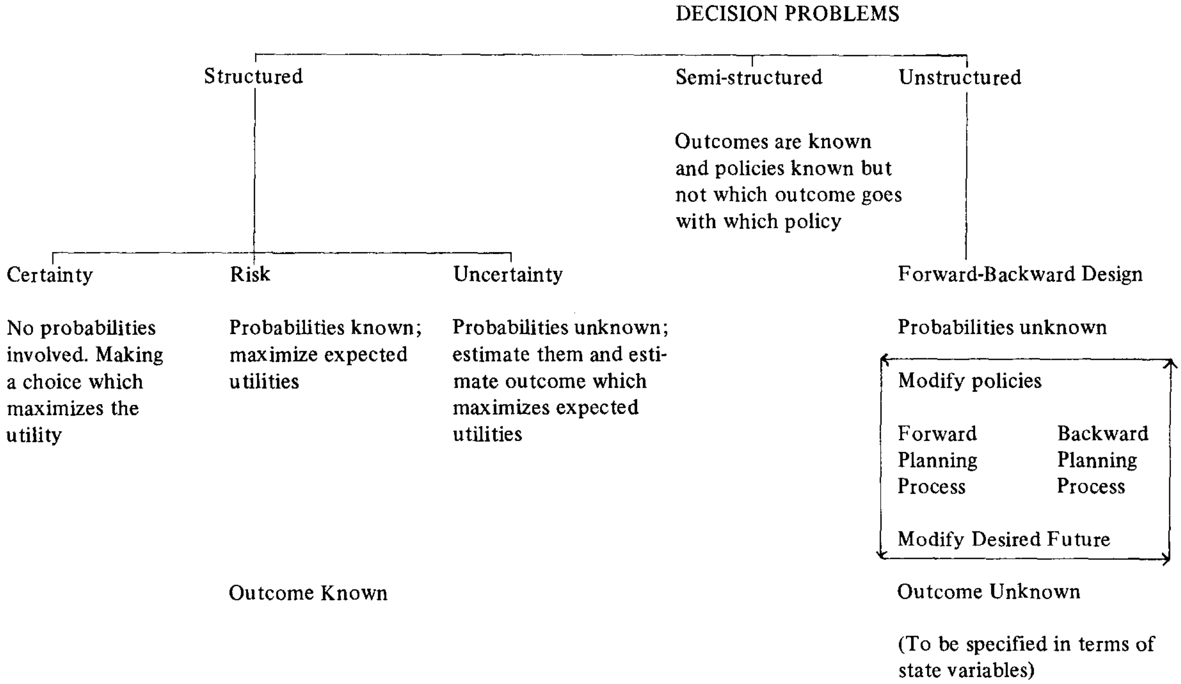

Fig. 1.

Our next problem is to identify a desired outcome for each of the actors and assist them to modify their objectives and allocate their effort to wield the necessary influence to overcome problems or utilize opportunities to attain their desired outcome. These objectives are adjoined to the forward process to test their effect on modifying the likely outcome more closely with the desired one. The process is repeated by modifying both the desired outcome and the change in objectives to attain that outcome.

2. Reciprocal matrices and ratio scale estimates

Assume that we are given n stones, $A_1, \ldots, A_n$, whose weights $w_1, \ldots, w_n$, respectively, are known to us. Let us form the matrix of pairwise ratios whose rows give the ratios of the weights of each stone with respect to all others. Thus we have the matrix A of table 1. We have multiplied A on the right by the vector of weights w. The result of this multiplication is nw. Thus, to recover the scale from the matrix of ratios we must solve the problem $Aw = nw$ or $(A - nI)\,w = 0$. This is a system of homogeneous linear equations. It has a nontrivial solution if and only if the determinant of $(A - nI)$ vanishes, i.e., n is an eigenvalue of A. Now A has unit rank since every row is a constant multiple of the first row. Thus all its eigenvalues except one are zero. The sum of the eigenvalues of a matrix is equal to its trace and in this case, the trace of A is equal to n. Thus n is an eigenvalue of A and we have a nontrivial solution. The solution consists of positive

Table 1
The matrix A

$$
A \equiv \begin{array}{c|cccc}
 & A_1 & A_2 & \cdots & A_n \\
\hline
A_1 & \frac{w_1}{w_1} & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_n} \\
A_2 & \frac{w_2}{w_1} & \frac{w_2}{w_2} & \cdots & \frac{w_2}{w_n} \\
\vdots & \vdots & \vdots & & \vdots \\
A_n & \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & \frac{w_n}{w_n}
\end{array}
\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = n \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}
$$

entries and is unique to within a multiplicative constant (by the Perron–Frobenius theorem since A is irreducible, i.e., it is not decomposable into blocks of the form:

$$A = \begin{pmatrix} C_1 & 0 \\ C_2 & C_3 \end{pmatrix}.$$

To make w unique we normalize its entries by dividing by their sum. Thus given the comparison matrix we can recover the scale. In this case the solution is any column of A normalized. Note that in A we have $a_{ji} = 1/a_{ij}$ the reciprocal property. Thus, also, $a_{ii} = 1$. Also, A is consistent, i.e., its entries satisfy the condition

$$a_{jk} = a_{ik}/a_{ij} .$$

Thus the entire matrix can be constructed from a set of n elements which form a chain across the rows and columns.

In the general case we cannot give the precise values of w_i/w_j but estimates of them. For the moment let us consider an estimate of these values by an expert who we assume makes small errors in judgment. From matrix theory we know that small perturbation of the coefficients implies small perturbation of the eigenvalues. Our problem now becomes $A'w' = \lambda_{\max} w'$ where $\lambda_{\max}$ is the largest eigenvalue of A'. To simplify the notation we shall continue to write $Aw = \lambda_{\max} w$ where A is the matrix of pairwise comparisons. The problem now is how good is the estimate w. Note that if we obtain w by solving this problem the matrix whose entries are w_i/w_j is a consistent matrix which is our consistent estimate of the matrix A. A itself need not be consistent. In fact, the entries of A need not even be ordinally consistent, i.e., A_1 may be preferred to A_2, A_2 to A_3, but A_3 is preferred to A_1. What we would like is a measure of the error due to inconsistency. It turns out that A is consistent *if and only if* $\lambda_{\max} = n$ and that we always have $\lambda_{\max} \geqslant n$. This suggests using $\lambda_{\max} - n$ as an index of departure from consistency. But

$$\lambda_{\max} - n = -\sum_{i=1}^{n} \lambda_i , \qquad \lambda_{\max} = \lambda_1 ,$$

where λ_i, $i = 1, ..., n$, are the eigenvalues of A. We adopt the average value $(\lambda_{\max} - n)/(n - 1)$ which is the (negative) average of λ_i, $i = 2, ..., n$ (some of which may be complex conjugates!). On calculating this value we compare the result with those of the same index obtained as an average over a large number of matrices of the same order whose entries are random. However, we preserve the relations $a_{ji} = 1/a_{ij}$, $a_{ii} = 1$ in these matrices to improve consistency. The reason for this is that if one stone is estimated to be k times heavier than another, it does not seem unreasonable to require that the second stone be estimated to be $1/k$ times the weight of the first. If the ratio of our index to that from random matrices is significantly small, we accept the estimates. Otherwise, we attempt to improve consistency by obtaining additional information. We shall not go into the details of this procedure.

It is interesting to note that $2(\lambda_{\max} - n)(n - 1)$ is the variance of the error incurred in estimating a_{ij}. This can be shown by writing:

$$a_{ij} = (w_i/w_j)\, \epsilon_{ij}, \qquad \epsilon_{ij} > 0$$

and

$$\epsilon_{ij} = 1 + \delta_{ij}, \qquad \delta_{ij} > -1.$$

It is δ_{ij} that concerns us as the error component and its value for an unbiased estimator, i.e., $|\delta_{ij}| < 1$.

What would be desirable is to obtain the density function of $(\lambda_{\max} - n)/(n - 1)$.

To conclude this section, we note that solution of the largest eigenvalue problem when normalized gives us a unique estimate of an underlying ratio scale.

Before we go on to discussion of what numerical values we might use in making comparisons we need to discuss the problem of the size of a matrix.

We note that $(n^2 - n)/2$ judgments or estimates are necessary to fill a matrix. The question is whether this can be done with fewer estimates still obtaining a good answer.

We note that in making the estimates and to keep the comparisons relevant an individual has to keep in mind all the elements being compared. It is known that an individual cannot simultaneously compare more than 7 ± 2 elements. If this is so, then how is it that we have measurement across wide classes of objects. The answer to this is by hierarchical decomposition. The elements are grouped ordinally (as a first estimate) into comparability classes of about seven elements each. The element with the highest weight in the class of lighter weight elements is also included in the next heavier class and serves as a pivot to uniformize the scale between the two classes. The procedure is repeated from a class to an adjacent one until we have all the elements appropriately scaled [3,4].

In passing, we note that the eigenvector approach to measurement (as one might expect) preserves ordinal preferences among the alternatives, i.e., if an alternative is preferred to another, its eigenvector component is larger than that of the other. This theorem is also true in the general hierarchical use of eigenvalues which we discuss next.

3. What is a hierarchy

Hierarchies play an important role in studying the impact of alternatives on the goals and objectives of a system. Usually the objectives are satisfied in different levels according to their generality and degree of importance. By imbedding a problem in hierarchical form, one has to identify all the important factors and the levels to which they belong and measure the interaction between levels. Finally, a principle of hierarchical composition is used to measure the impact of the factors in the lowest level on the highest objective of the hierarchy.

There are several kinds of hierarchies, the simplest of which are dominance hierarchies which descend like an inverted tree with the boss at the top, followed by successive levels of bossing. Another kind are holarchies which are essentially dominance hierarchies with feedback. Chinese box (or modular) hierarchies grow in size from the simplest elements or components (the inner boxes) to larger and larger aggregates (the outer boxes). In biology, neogenetic hierarchies are of interest because they have successive newly emerging top levels through evolution. We shall concentrate our attention on dominance hierarchies, although the theory described below is being generalized to the other hierarchical forms.

There are many ways of defining a hierarchy. The one which suits our needs best here is the following: In an ordered set, we define $x < y$ to mean that $x \leqslant y$ and $x \neq y$. y is said to cover (dominate) x if $x < y$ and if $x < t < y$ is possible for no t. We used the notation $x^- = \{y \mid x \text{ covers } y\}$ and $x^+ = \{y \mid y \text{ covers } x\}$, for any element x in an ordered set.

Definition. Let H be a finite partially ordered set. H is a hierarchy if it satisfies the conditions:

a) There is a partition of H into sets L_k, $k = 1, ..., h$ where $L_1 = \{b\}$, b a single element.
b) $x \in L_k$ implies $x^- \subset L_{k+1}$, $k = 1, ..., h - 1$.
c) $x \in L_k$ implies $x^+ \subset L_{k-1}$, $k = 2, ..., h$.

For each $x \in H$, there is a suitable weighting function (whose nature depends on the phenomenon being hierachically structured):

$$w_x : x^- \; [0, 1] \text{ such that } \sum_{y \in x^-} w_x(y) = 1.$$

The sets L_i are the levels of the hierarchy, and the function w_x is the priority function of the elements in one level with respect to the objective x. We observe that even if $x^- \neq L_k$ (for some level L_k), w_x may be defined for all of L_k by setting it equal to zero for all elements in L_k not in x^-.

The weighting function, we feel, is a significant contribution towards the application of hierarchy theory.

Definition. A hierarchy is complete if, for all $x \in L_k$, $x^+ = L_{k-1}$, for $k = 2, ..., h$.

We can state the central question:

Basic problem. Given any element $x \in L_\alpha$, and subset $S \in L_\beta$, $(\alpha < \beta)$, how do we define a function $w_{x,S}$: $S \to [0, 1]$ which reflects the properties of the priority functions w_y on the levels L_k, $k = \alpha, ..., \beta - 1$. Specifically, what is the function $w_{b, L_h} : L_h \to [0, 1]$?

In less technical terms, this can be paraphrased thus: Given a social (or economic) system with a major objective b, and the set L_h of basic activities, such that the system can be modelled as a hierarchy with largest element b and lowest level L_h. What are the priorities of the elements of L_h with respect to b?

From the standpoint of optimization, to allocate a resource among the elements any interdependence must also be considered. Analytically, interdependence may take the form of input-output relations such as, for example, the interflow of products between industries. A high priority industry may depend on flow of material from a low priority industry. In an optimization framework, the priority of the elements enables one to define the objective function to be maximized, and other hierarchies supply information regarding constraints, e.g., input-output relations.

We shall now present our method to solve the basic problem. Assume that $Y = \{y_1, ..., y_{m_k}\} \in L_k$ and that $X = \{x_1, ..., x_{m_{k+1}}\} \in L_{k+1}$. Without loss of generality we may assume that $X = L_{k+1}$, and that there is an element $z \in L$ such that $y \subset z^-$. We then consider the priority functions

$$w_z : Y \to [0, 1]$$

and

$$w_y : X \to [0, 1], \qquad j = 1, ..., n_k.$$

We construct the "priority function of the elements in X with respect to z," denoted w, $w : X \to [0, 1]$, by

$$w(x_i) = \sum_{j=1}^{n_k} w_{y_j}(x_i)\, w_z(y_j)\ , i = 1, ..., n_{k+1}.$$

It is obvious that this is no more than the process of weighting the influence of the element y_j on the priority of x_i by multiplying it with the importance of x_i with respect to z.

The algorithms involved will be simplified if one combines the $w_{y_j}(x_i)$ into a matrix B by setting $b_{ij} = w_{y_j}(x_i)$. If we further set $w_i = w(x_i)$ and $w_j = w_z(y_j)$, then the above formula becomes

$$w_i = \sum_{j=1}^{n_k} b_{ij} w_j', \qquad i = 1, ..., n_{k+1}.$$

Thus, we may speak of the priority vector w and, indeed, of the priority matrix B of the $(k + 1)$st level; this gives the final formulation

$$w = Bw' .$$

The following is easy to prove:

Theorem. *Let H be a complete hierarchy with largest element b and h levels. Let B_k be the priority matrix of the kth level, $k = 1, ..., h$. If w' is the priority vector of the pth level with respect to some element z in the $(p - 1)$st level, then the priority vector w of the qth level $(p < q)$ with respect to z is given by*

$$w = B_q B_{q-1} \cdots B_{p+1} w'.$$

Thus, the priority vector of the lowest level with respect to the element b is given by:

$$w = B_h B_{h-1} \cdots B_2 b_1 .$$

If L_1 has a single element, $b_1 = 1$. Otherwise, b_1 is a prescribed vector.

The following observation holds for a complete hierarchy but it is also useful in general. The priority of an element in a level is the sum of its priorities in each of the comparison subsets to which it belongs; each weighted by the fraction of elements of the level which belong to that subset and by the priority of that subset. The resulting set of priorities of the elements in the level is then normalized by dividing by its sum. The priority of a subset in a level is equal to the priority of the dominating element in the next level.

4. The scale

The scale we recommend for use which has been successfully tested and compared with other scales will now be discussed.

The judgments elicited from people are taken qualitatively and corresponding scale values assigned to them. In general, we do not expect the judgments to be consistent.

Our choice of scale hinges on the following observation. Roughly, the scale should satisfy the requirements:

1) It should be possible to represent people's differences in feelings when they make comparisons. It should represent as much as possible all distinct shades of feeling that people have.
2) If we denote the scale values by $x_1, x_2, ..., x_p$, then let $x_{i+1} - x_i = 1, ..., p - 1$.

Since we require that the subject must be aware of all gradations at the same time, and we agree with the psychological experiments [3] which show that an individual cannot simultaneously compare more than seven objects (plus or minus two) without being confused, we are led to choose a $p = 7 + 2$. Using a unit difference between successive scale values is all that we allow, and using the fact that $x_1 = 1$ for the identity comparison, it follows that the scale values will range from one to nine.

As a preliminary step towards the construction of an intensity scale of importance for activities, we have broken down the importance ranks as shown in the scale of table 2. In using this scale the reader should recall that we assume that the individual providing the judgment has knowledge about the relative values of the elements being compared whose ratio is $\geqslant 1$ and that the numerical ratios he forms are nearest integer

Table 2
The scale and its description

Intensity of importance	Definition	Explanation
1 [a]	Equal importance	Two activities contribute equally to the objective
3	Weak importance of one over another	Experience and judgment slightly favor one activity over another
5	Essential or strong importance	Experience and judgment strongly favor one activity over another
7	Demonstrated importance	An activity is strongly favored and its dominance demonstrated in practice
9	Absolute importance	The evidence favoring one activity over another is of the highest possible order of affirmation
2, 4, 6, 8	Intermediate values between the two adjacent judgments	When compromise is needed
Reciprocals of above non-zero	If activity i has one of the above non-zero numbers assigned to it when compared with activity j, then j has the reciprocal value when compared with i	
Rationals	Ratios arising from the scale	If consistency were to be forced by obtaining n numerical values to span the matrix

[a] On occasion in 2 by 2 problems, we have used $1 + \epsilon$, $0 < \epsilon < \frac{1}{2}$ to indicate very slight dominance between two nearly equal activities.

approximations scaled in such a way that the highest ratio corresponds to nine.

At first glance one would like to have a scale extend as far out as possible. On second thought we discover that to give an idea how large measurement can get, scales must be finite. We also note that one does not measure widely disparate objects by the same yardstick. Short distances on a piece of paper are measured in centimeters, longer distances in a neighborhood in meters, and still larger ones in kilometers and even in light years. To make comparisons of the sizes of atoms with those of stars, people, in a natural fashion, insert between these extremes, objects which gradually grow larger and larger enabling one to discriminate in the process of transition among the orders of magnitude of measurement. To make such distinction possible the objects put in each group are within the range of the scale and the largest object in one group is used as the smallest one in the next larger group. Its scale values in the two groups enable one to continue the measurement from one group to the next and so on. In practice, one way or another, the numerical judgments will have to be approximations, but how good is the question at which our theory is aimed.

A typical question to ask in order to fill in the entries in a matrix of comparisons is: Consider two properties i on the left side of the matrix and another j on the top; which of the two has the property under discussion more, and how strongly more (using the scale values 1 to 9)? This gives us a_{ij}. The reciprocal value is then automatically entered for a_{ji}.

Yet there is no satisfactory statistical theory that would assist us in deciding how well judgmental data correspond to reality. We have occasionally used the root mean square deviation (RMS) and the median absolute deviation about the median (MAD). These indicators are probably more useful when making inter-scale or inter-personal comparisons in judgments than as absolute measures of the goodness of fit. We have not found the Chi-square test useful. It is clear that this is an area of research that is worth pursuing.

Considerable effort has been concentrated on comparing the scale 1 to 9 with 25 other scales suggested to us by a number of people. Space limitation prevents us from showing that this scale and small perturbations of it are better than practically all others. We used five different problems for which the real answers were later determined and the root mean square deviation and the median absolute deviation about the median were used for comparison of goodness of fit of the resulting eigenvector.

5. Examples

An interesting example illustrating the fact that eigenvectors give the correct results in the field of probability, consider an urn with balls of three colors: two black, one white and three red. The probabilities of drawing a ball of one of these colors are, respectively, $\frac{2}{6}, \frac{1}{6}, \frac{3}{6}$.

The pairwise comparison matrix giving the ratios of the number of balls of indicated colors is given in table 3. Since this matrix is consistent, the entries of any column after normalizing gives the ratio of balls of each color to the total. We have, for example, from the first column, $\frac{2}{6}, \frac{1}{6}, \frac{3}{6}$ which, as to be expected, corresponds to the probabilities that a ball drawn at random has one of the three colors. If the urn has a large number of balls whose relative quantities can be compared, this approach would give an estimate of the relative magnitude of each kind.

Table 3

	B	W	R
B	1	2	$\frac{2}{3}$
W	$\frac{1}{2}$	1	$\frac{1}{3}$
R	$\frac{3}{2}$	3	1

5.1. *The consumption of drinks in the United States*

Three individuals were asked to compare seven drinks consumed extensively in the United States with the idea that their estimates first done individually and then collectively of the dominance of consumption may give a close idea of the actual percentages among the seven drinks. The matrices are given in table 4. The actual percentages of drinks consumed, eigenvectors and eigenvalues are given in table 5.

It should be emphasized that the individuals did not have access to the actual consumption figures until after the completion of the exercise. In all cases the closeness of the results was impressive and the consistency not bad.

It is interesting to note that a group may not do better than an individual if some of them are more articulate than others and in addition have strong biases. It is best that they be allowed to speak briefly without forcing the vote their way. Generally, people should be encouraged to stick to their feelings unless they get a convincing and overriding reason from the others to change their beliefs. An average is probably better than a group decision under "no holds barred" type of coercion. On the other hand, fair democratic kind of interaction has often led to excellent results better than what individuals could do.

5.2. *Choosing a job*

A student who had just received his Ph.D. was interviewed for three jobs. His criteria for selecting the jobs and their pairwise comparison matrix are given in table 6.

The pairwise comparison matrices of the jobs with respect to each criterion are given in table 7. The eigenvalue and eigenvector of the first matrix are, respectively: $\lambda_{max} = 6.35$; 0.16, 0.19, 0.19, 0.05, 0.12, 0.30.

The remaining eigenvalues and eigenvectors are given in table 8. The composite vector for the jobs is given by:

$$A = 0.40, \qquad B = 0.34, \qquad C = 0.26.$$

The differences were sufficiently large for the candidate to accept the offer of job A.

5.3. *Beverage container problem*

Seven types of containers made of glass, bimetallic and aluminium cans to be used by the beverage industry were evaluated based on four criteria: energy-consumption, cost, environmental waste and customer convenience.

The container types were: 1) refillable glass, no recycle (GRNR); 2) refillable glass, recycle (GRR); 3) throwaway glass, no recycle (GTNR); 4) throwaway glass, recycle (GTR); 5) bimetallic can, no recycle

Table 4

Individual A

	Coffee	Wine	Tea	Beer	Soft drinks	Milk	Water
Coffee	1	5	7	2	$\frac{1}{3}$	2	$\frac{1}{7}$
Wine		1	1	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{5}$	$\frac{1}{7}$
Tea			1	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{9}$
Beer				1	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{5}$
Soft drinks	(Reciprocals)				1	$\frac{1}{2}$	$\frac{1}{4}$
Milk						1	$\frac{1}{3}$
Water							1

Individual B, C

Coffee	1, 1	9, 7	2, 7	4, 5	3, 3	1, 3	$\frac{1}{3}$, $\frac{1}{3}$
Wine		1, 1	$\frac{1}{8}$, 2	$\frac{1}{5}$, $\frac{1}{5}$	$\frac{1}{7}$, $\frac{1}{5}$	$\frac{1}{8}$, 1	$\frac{1}{7}$, $\frac{1}{7}$
Tea			1, 1	3, $\frac{1}{3}$	1, $\frac{1}{3}$	$\frac{1}{4}$, $\frac{1}{2}$	$\frac{1}{3}$, $\frac{1}{7}$
Beer				1, 1	2, $\frac{1}{5}$	$\frac{1}{5}$, 1	$\frac{1}{4}$, $\frac{1}{7}$
Soft drinks	(Reciprocals)				1, 1	$\frac{1}{5}$, 2	$\frac{1}{3}$, 1
Milk						1, 1	$\frac{1}{2}$, 9
Water							1, 1

Group

Coffee	1	9	5	4	2	3	$\frac{1}{3}$
Wine		1	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{9}$
Tea			1	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{7}$
Beer				1	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{5}$
Soft drinks	(Reciprocals)				1	3	$\frac{1}{3}$
Milk						1	$\frac{1}{7}$
Water							1

Table 5

	Actual	A	B	C	Average of A, B, C	Group
Coffee	0.20	0.13	0.19	0.25	0.19	0.22
Wine	0.01	0.03	0.02	0.04	0.03	0.02
Tea	0.04	0.02	0.10	0.03	0.05	0.05
Beer	0.12	0.08	0.07	0.07	0.07	0.06
Soft drinks	0.18	0.17	0.07	0.18	0.14	0.13
Milk	0.14	0.15	0.25	0.06	0.15	0.09
Water	0.30	0.42	0.30	0.37	0.36	0.42
λ_{max}		7.82	7.62	7.77		7.56

Table 6
Overall satisfaction with job

	Research	Growth	Benefits	Colleagues	Location	Reputation
Research	1	1	1	4	1	$\frac{1}{2}$
Growth	1	1	2	4	1	$\frac{1}{2}$
Benefits	1	$\frac{1}{2}$	1	5	3	$\frac{1}{2}$
Colleagues	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{5}$	1	$\frac{1}{3}$	$\frac{1}{3}$
Location	1	1	$\frac{1}{3}$	3	1	1
Reputation	2	2	2	3	3	1

Table 7

Research

	A	B	C
A	1	$\frac{1}{4}$	$\frac{1}{2}$
B	4	1	3
C	2	$\frac{1}{3}$	1

Growth

	A	B	C
A	1	$\frac{1}{4}$	$\frac{1}{5}$
B	4	1	$\frac{1}{2}$
C	5	2	1

Benefits

	A	B	C
A	1	3	$\frac{1}{3}$
B	$\frac{1}{3}$	1	1
C	3	1	1

Colleagues

	A	B	C
A	1	$\frac{1}{3}$	5
B	3	1	7
C	$\frac{1}{5}$	$\frac{1}{7}$	1

Location

	A	B	C
A	1	1	7
B	1	1	7
C	$\frac{1}{7}$	$\frac{1}{7}$	1

Reputation

	A	B	C
A	1	7	9
B	$\frac{1}{7}$	1	5
C	$\frac{1}{9}$	$\frac{1}{5}$	1

Table 8

	Research	Growth	Benefits	Colleagues	Location	Reputation
λ_{max} =	3.02	3.02	3.56	3.06	3	3.21
Company A	0.14	0.10	0.32	0.28	0.47	0.77
Company B	0.63	0.33	0.22	0.65	0.47	0.17
Company C	0.24	0.57	0.46	0.07	0.07	0.05

Table 9

	Energy	Cost	Environmental waste	Customer convenience
Energy	1	5	3	9
Cost	$\frac{1}{5}$	1	$\frac{1}{4}$	8
Environmental waste	$\frac{1}{3}$	4	1	9
Customer convenience	$\frac{1}{9}$	$\frac{1}{8}$	$\frac{1}{9}$	1

Table 10

GRNR	GRR	GTNR	GTR	BMNR	ALNR	ALR
0.31831	0.31831	0.09529	0.10303	0.10683	0.02737	0.03086

Table 11

GRNR	GRR	GTNR	GTR	BMNR	ALNR	ALR
0.32	0.30229	0.09335	0.09318	0.08394	0.05224	0.0550

(BMNR); 6) aluminium can, no recycle (ALNR); 7) aluminium can, recycle (ALR).

The judgmental matrix of the pairwise comparison of the four objective factors is given in table 9. The containers were then compared with respect to each criterion. The composite weight vector is given in table 10. It is interesting to note that the above results were consistent when, instead of the judgmental data of pairwise comparison, the actual data based on published literature pertaining to energy, cost and environmental waste were used. The matrices for the fourth criterion – customer convenience and the weighting matrix of the objective criterion – were taken as in the above case. This yielded the priorities of table 11 by way of validation which is close to the previous vector.

It needs to be pointed out that the quantitative factors had minimum and maximum values attached to them which served as indicators for the range of values of the 1–9 scale used together with an idea of the strength of utility. In any case, glass containers are favored in the analysis corresponding to their increased use in practice.

6. Planning

We now turn to a brief verbal description of an important use of the method. Regrettably at this point our exposition has surpassed the space allocated for this article. For further detail see [2,6].

Planning is a process concerned with ends and means or outcomes and policies to attain them. It is a two point boundary value process. One boundary point is fixed at the present and defined by the actors or stakeholders, their objectives, policies and outcomes arising out of these policies. This is the *forward planning process* (a descriptive process) concerned with the assessment of the beliefs of the actors as to which is the most likely outcome. The other boundary point of planning is fixed at the future and defined by the desired outcome, the actors who wield the greatest influence (or obstacles) in attaining that future, the objectives of these actors and specific policies which a given actor may pursue in light of these (other actor) objectives to induce changes which lead to the desired future. This is the *backward planning process* (a normative process) concerned with the assessment of what is the best way to attain a desired outcome. The backward process is evaluated for each actor separately and the new policies for all actors are adjoined to the forward process to test their effectiveness when all actors pursue their objectives simultaneously. The resulting forward outcome is determined and compared with the desired outcome for each actor. The desired outcomes and their corresponding policies are revised and the procedure of testing their effectiveness on the forward planning process is repeated. In this manner the forward and backward processes are "aligned" for closer results between what is desired by each actor and what will be the actual outcome.

The technical details of using the hierarchical approach to planning have been illustrated in a forward planning process to estimate the future of higher education in the U.S. in 1985–2000 [6]. The actors were students, faculty, administrative, government, private donors and industry. Each actor has several objectives and the projected outcome is obtained through a weighting process of a number of contrasting outcomes. All outcomes are characterized by a set of state variables whose values are estimated on a difference scale according to how they differ from the present taken as the origin of reference. The corresponding values for the variables of the composite (projected) outcome are calculated as convex combinations of those of the contrasting scenarios using the hierarchical approach to first weight the actors impact on higher education, then weight (prioritize) the objectives of each actor and then prioritize the outcomes as they affect the objectives of the actors and finally derive the composite weights or impacts of the outcomes on the future of higher education.

The backward planning process has been used to design a transport system for the Sudan, an agriculturally rich country that has been singled out as a potential breadbasket for several hundred million people in Africa and the Middle East. Here a composite desired future was constructed from a set of contrasting likely futures. The impact of the regions of the country on this future and the impact of 103 transportation projects on their corresponding regions were derived. The resulting priorities of the projects were of great value for implementation purposes. Social and political factors were used along with economics.

The joint forward-backward planning process has been carried out in an application made to resolve the conflict in Northern Ireland involving information and interaction with parties knowledgeable and involved in the conflict. An interesting result of this analysis was the emergence of a dominion status rule for Northern Ireland – an outcome that turns out to be the most stable – given the parties and their objectives [1].

The limited amount of space allotted to this article does not permit us to give details, but we hope that the interest of the reader has been adequately stimulated to pursue the ideas through the references. The method of analytical hierarchies offers an effective and realistic approach to prioritization, systems planning and to conflict resolution. Nearly 30 applications of the theory have been made in areas ranging from energy and mineral resources to the assessment of social and political influence.

The method has been generalized from hierarchies to systems with feedback given in the form of networks.

7. Conclusion

There is a considerable interest today in the question as to whose judgments should be used in a planning process. It is precisely here that the forward and backward processes have proved to be of immense value both in the presence and absence of participation. In the forward process we include the judgment of each actor as it relates to his areas of expertise and thus obtain a composite outcome which reflects the judgments of all the actors. If the actors wish to cooperate and justify their judgments to each other through debate and consensus one can use their collective judgment across all levels of the hierarchy; but as we have just said, this is not essential.

In the backward process we use the desired preferences of each actor individually. The iterative process should serve to educate the actors as to the effectiveness of their policies without due consideration to other actor influences since the outcome is a resultant of these influences.

Finally, there are cases where unless one were to give all the actors equal weights (which is rarely justified in practice) it is essential to establish relevant (objective) criteria for this weighting. For example, their years of experience, degree of success measured in terms of concrete accomplishments or, in general, whatever the actors may agree (or disagree) on as grounds which serve to show their relative strengths can be used for this purpose.

Sometimes it may be desired to analyze a problem completely from the standpoint of each actor. In that case, the outcomes could be synthesized by using an appropriate weighting of the actors to obtain a resultant outcome. The method of analytical hierarchies has also served very well in facing these questions. In a second forthcoming paper we propose to develop several of the foregoing ideas with additional illustrations.

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