

APPLICATIONS OF ANALYTICAL HIERARCHIES

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1. Introduction

In a previous article which appeared in this journal [16] I outlined some of the ideas involved in measuring priorities in hierarchical systems. It was my hope to show that when dealing with complexity, a hierarchical structure with ratio scale measurement provides a natural expression of the mind to deal with the detail of a problem. From the numerous varied applications we have made using these ideas, it would seem that most problem areas particularly those of concern to decision making can be dealt with through the theory of analytical hierarchies, when necessary incorporating optimization techniques and probabilities. We have also generalized the theory of analytical hierarchies to network systems with feedback.

A crucial area that remains open when modelling problems in the framework of hierarchies is the relation of space and time to hierarchical thinking.

Perhaps one of the greatest uses we can make of any theory is to improve our understanding of whatever we wish to study and to enable us to predict (by projecting our beliefs) what might happen and thus provide us with a capacity for decision and control.

In the first paper we showed how analytical hierarchies are a reflection of the way our mind works through pairwise comparisons and through clustering to deal with complexity. We also showed that the eigenvalue approach to ratio scaling enables one to scale qualitative social values and make tradeoffs between criteria some of which we know how to measure and others for which so far we have no convenient measurement.

Here we shall give examples of prediction (and hindsight). In the decision and control area we are usually interested in problems of priorities (focusing on highest priority alternatives), resource allocation, optimization, conflict resolution, planning for the

future, determining requirements, designing systems, measuring their performance, improving their reliability, and maintaining their stability of operation. The study of each of these areas (most of which have already been investigated in some detail) requires a separate paper. Thus we shall be content to give brief summaries of some examples.

2. Design of a transport system for The Sudan – Priorities – Investment [5,6]

Against a background of great potential agricultural riches, the Sudan, the largest country in Africa (967,491 sq. mi.) but with only an 18.2 million population, is today a poor country with a GNP of about 2.8 billion dollars. Oil countries in the Middle East and international agencies, including the World Bank, recognize the capacity of the Sudan as a major provider of food for Africa and the Middle East, and have been investing in its development.

Incidentally, the oil rich Arab countries' populations do not exceed 20 million and, hence, their need does not even begin to make a difference in how many people the Sudan can feed. Even if its northern neighbor, Egypt, were to be included for one half of its population (estimated at 50 million by 1985) to be fed by Sudan, there would still be land to feed perhaps one hundred million more people. The entire economy of the Sudan and, in particular, the agricultural sector suffer from lack of adequate transportation.

The Sudan is serviced by four major modes of transportation: rail, road, river and air. These modes are combined together to provide a sparse and far-flung transportation infrastructure. The air network is centered at Khartoum and the rail and road systems are oriented for export through Port Sudan. The

The functions of a system can be represented by a hierarchy with the most important "driving" purposes occupying the top level and the actual operations of the system at the lowest level. In the case of the Sudan, overall development occupied the top level, followed by a level of scenarios or feasible outcomes of the future. The third level consisted of the regions of the Sudan as it was desirable to know their impact on the scenarios and in turn the impact of the scenarios on the top level. The fourth level consisted of the projects for which it is desired to establish their priorities by studying their impacts on the regions. Thus, we had a four level hierarchy in the study. By composing the impacts of the fourth level on the third, the third on the second and the second on the first, we obtain the overall impact of each project in the fourth level on the overall development of the Sudan represented in the first.

Pairwise comparison of the four scenarios according to their feasibility and desirability by 1985 gave rise to the matrix presented in Table 1. The priorities of the scenarios in the order they are listed are: 0.05, 0.61, 0.25, 0.09. As can be seen, Scenario II dominates, with Scenario III next in importance. Since the future is likely to be neither one nor the other, but rather a composition of these scenarios — with emphasis indicated by the priorities — this information was used to construct a composite scenario of the Sudan of 1985. This scenario is intended as the anticipated actual state of the future, it being a proportionate mix of the forces which make up the four scenarios described above. The Composite Scenario takes the main thrust of Scenario II, the future given by far the

	I	II	III	IV	
Status Quo	I	1	1/7	1/5	1/3
Agricultural Export	II	7	1	5	5
Balanced Regional Growth	III	5	1/5	1	5
Arab-African Regional Expansion	IV	3	1/5	1/5	1

Table 2
Priority weights of regions (percent)

	Bahr El Ghazal	Blue Nile	Darfur	East Equatoria	Gezira	Kassala	Khartoum	Kordofan	Northern	Red Sea	Upper Nile	West Equatoria
	3.14	6.55	5.37	1.70	12.41	5.25	21.40	5.96	2.94	22.54	3.37	9.39

highest priority, and is enlarged and balanced with certain elements from Scenarios III and IV. This composition indicates the likelihood of a synergistic amplification of individual features.

The Sudan has 12 regions whose individual economic and geographic identity more or less justifies political division into distinct entities. The regions were compared pairwise in separate matrices according to their impact on each of the scenarios. They comprise the third hierarchy level. The resulting eigenvectors are used as the columns of a matrix which, when multiplied by the eigenvector of weights or

priorities of the scenarios, gave a weighted average for the impact of the regions. We have then Table 2.

Now the projects of which there were 103 determined according to GNP growth rates which suggest supply-demand and flow of goods, comprised the fourth level of the hierarchy. They were compared pairwise in 12 matrices according to their impact on the regions to which they physically belonged. A project may belong to several regions and this had to be considered. The resulting matrix of eigenvectors was again weighted by the vector of regional weights to obtain a measure of the overall impact of each project

Table 3

The transportation development plan: phase I
(1974 Price Level in LS * 000,000; LS ≡ \$ 2.50)
(6% GNP GROWTH RATE)

Projects	Distance (km.)	Priority	CLASS G.N.P. RATES						COST		
			4.3%		6.0%		7.3%		A	B	C
			L	H	L	H	L	H			
RAIL											
Port Sudan–Haiya	203	4.724	A	B	A	B	A	B	9.10	7.10	–
Haiya–Atbara	271	3.455	B	B	B	B	A	B	12.20	9.50	–
Atbara–Khartoum	313	8.443	B	B	B	B	A	B	14.10	11.00	–
El Rahad–Babanusa	363	1.005	B	B	B	B	B	B	–	12.70	–
Fleet (6% GNP)											
Maintenance facilities											
Sub-total											
ROAD											
Wad Medani–Gedaref	231	2.840	A	A	A	A	A	A	23.90	–	–
Gedaref–Kassala	218	0.872	A	A	A	A	A	A	14.20	–	–
Kassala–Haiya–Port Sudan	625	2.229	A	A	A	A	A	A	50.00	–	–
Wad Medani–Sennar	100	0.526	A	A	A	A	A	A	14.90	–	–
Sennar–Kosti	110	0.345	A	A	A	A	A	A	7.20	–	–
Sennar–Es Suki	47	0.546	A	A	A	A	A	A	7.00	–	–
Ed Dubeibat–Kadugli	137	1.253	C	C	C	C	B	C	–	12.30	8.80
Kadugli–Talodi	100	0.266	–	–	–	–	B	–	–	6.60	–
Nyala–Kass–Zalingei	210	0.951	B	C	B	C	B	C	–	11.30	7.40
Jebel Al Aulia–Kosti *	300	1.567	B	B	B	B	A	B	44.70	29.70	–
Juba–Nimuli	190	0.329	C	C	C	C	B	C	–	8.70	5.30
Juba–Amadi–Rumbek–Wau	725	0.494	C	C	C	C	C	C	–	–	20.30
Fleet											
Sub-total											

* The priority rating of this project is based mostly on potential rather than present development. In view of its high cost relative to other road projects, it has been omitted. It is recommended that it be given urgent consideration in the following planning period.

on the future. This gave rise to the kind of table of which there are nine in the final report: Table 3.

We examined the 4.3%, the present growth rate, and found that most of the current facilities with the prevailing level of efficiency would be crammed to their limit. Obviously, a compromise with a rational justification for growth had to be made somewhere between these two extremes. When we examined the 6% GNP growth rate, found feasible by the econometric analysis, it provided excellent guidelines for those projects which were found to be needed at 4.3% and remained invariant with high priority at 7.3%. These

were mostly the projects we recommended for implementation. The ratios of priorities to costs served as a measure of the effectiveness of investment. Six billion dollars have been earmarked for expenditure in the Sudan over the next few years closely following some of the recommendations of the study.

Total investment requirements to achieve the Composite Scenario projected growth of real GNP at 4.3, 6 and 7.3% per year are given below in Table 4. For example, at 7.3% they are estimated to be approximately \$5105 million at 1974 price levels, or \$7647 million at current price levels (considering inflation

Recommended class	Main reason		Committed (financing in progress)	COST		
	Flow	Other		Total	Foreign currency	Local currency
A	X			9.10	4.55	4.55
B	X			9.50	6.30	3.20
B	X			11.00	7.30	3.70
B	X			12.70	8.50	4.20
				10.90	40.90	—
				2.00	1.00	1.00
				85.20	68.55	16.65
A	X		X	23.90	16.70	7.20
A	X		X	14.20	9.90	4.30
A	X		X	50.00	35.00	15.00
A	X		X	14.90	10.40	4.50
A	X		X	7.20	5.00	2.20
A	X			7.00	4.90	2.10
B	X		X	12.30	7.40	4.90
—						
B	X			11.30	6.80	4.50
—	X	High cost, alternative provided				
X	X			5.30	1.60	3.70
C	X	Together with alternate, Hgh. priority		20.30	6.10	14.20
				20.80	20.80	
				187.20	124.60	62.60

Table 4
Dollars (millions), current prices

	4.3%	6%	7.3%
Transport	978.33	1789.88	2899.96
Agriculture	1372.75	1695.53	2183.30
Industry	588.28	963.33	1456.08
Services	1307.35	1194.58	1107.20
Total	4246.71	5643.32	7646.54

between 1974 and 1985). The latter figure represents approximately 10% of the GNP each year over the planning period, 1972–1985. This will be divided among the major sectors as shown.

3. Input–output analysis by the eigenvalue method: construction of constraint

The method has been used to construct economic input–output tables proceeding in two steps [13]. The first step utilizes judgments to determine the relative impact of the different sectors on the economy. This is essentially an “a priori” or constant value of the sectors. The second step involves the analysis of the inter-dependence among the sectors. We take each sector and determine the relative strength of utilization of its output by the remaining sectors. The second step is a calculation of the current value of a sector in terms of its influence on the remaining sectors.

Finally we compose the results of the two steps to obtain the matrix of input coefficients. In cases where adequate information is not available for making overall comparisons, additional levels may be introduced in the hierarchy to facilitate the distribution of impacts with respect to subcriteria. For example, the overall impact of the sectors may be broken down into impact on utilization of capital, labor, demand, costs and so on. These criteria would then have to be compared according to their relative strength of impact on the economy and hierarchical weighting used to obtain their overall importance.

A part of the Sudan study involved the construction of econometric models with an input–output table by our colleague L.R. Klein at the Wharton Economic Forecasting Associates. This particular input–output table was developed on the basis of

information from surrounding countries and not directly from Sudanese data. Thus it is an indirect estimate. We used our procedure to obtain an input coefficients table based on qualitative information between the economic sectors of the Sudan. Since the number of sectors is small, it serves as a good short illustration of the method. Similar applications have been made to Pakistan and Iran with good results.

The outcome has been sufficiently striking that the method is now being used to construct input–output tables for social interactions – an idea entertained by Leontief himself in 1956 [20], but not pursued because of lack of a theory of measurement in social areas. It has been pointed out that the method has great potential application to areas where prices are not known but qualitative information about an economy is available, e.g. the USSR. It would require considerable space to give the details of an application.

4. The faculty tenure problem: determination of requirements [14]

Committees govern universities and among the most important (and perhaps fearsome) are those responsible for questions of tenure and the promotion and appointment of individuals to the ranks of professor and associate professor. Hopefully, the appointments reflect by and large the merit of the candidates. However, it is inevitable that in subtle ways they are also influenced by the likes and dislikes of the committee members. In general, however, unbiased one may try to be, it is difficult to achieve the kind of objectivity and high degree of impartiality demanded of persons passing judgment on their peers.

As a contribution towards improving this kind of decision making and making explicit as far as possible its subjective and objective aspects for easier interaction among the committee members, we examine the problem here and give a method for scaling the relative importance of the selection criteria, evaluating the candidates by these criteria and obtaining a measure of their desirability for the job.

Most of us are taught early in life that one cannot compare apples and oranges. Nevertheless, we are constantly comparing and implicitly indicating preferences among things like apples and oranges, making our

choices and altering reality. Is there an underlying rationality to this process? What does our mind do when it deals with complex phenomena? Its choices certainly do not seem to be capricious and random. Frequently it settles on what appears to be a good choice. We aspire to assist people to attain a clearer identification of the criteria and issues before them and a better assessment of the relative preferences among these criteria.

We begin by noting that people's requirements for a candidate for a senior position are generally more demanding than those for a lesser position. For example, one does not expect a young individual aspiring for the post of associate professor to have produced a powerful, deep theory with wide recognition as one might expect of an older, better seasoned person seeking a professorial position. Thus, even though the criteria for excellence may be the same, a committee's prescribed norms may differ in placing emphasis on these criteria depending on the seniority or distinction of the position sought.

Experience on such committees and a survey of the literature suggest the following criteria for evaluation. We note that these criteria fall into two broad classes. The first relates to a candidate's research work and the second, to his or her teaching ability. Now, a candidate's research work may be evaluated on the basis of a number of criteria. We have the significance of the candidate's work; the quality and integrity indicated in the writings; provocativeness of thought; the variety of ideas and their growth over the years; the numbers of papers published and the general reputation of the candidate in the field. Similarly, a candidate's teaching ability may be evaluated on the basis of scholarship and depth of knowledge; ability to coherently express oneself; openness to new ideas; commitment to the subject; relationship with students in terms of accessibility; influence on their approach to the subject; ability to encourage the weaker students and develop the critical abilities of the brighter students. Different committees may have additional and different criteria with varying emphasis on each but the basic approach would be the same. They may add other contributions to the university as criteria. In addition, departments may have a portfolio of differing requirements such as teaching capability from some and research capabilities from others.

We now obtain relative weights for the criteria as

judged by the committee.

To derive such weighting, we first make pairwise comparisons among the criteria relating to a candidate's research work indicating the strength of preference we have for one over each of the others with respect to its contribution to distinction in research. We separately do the same for the criteria relating to teaching. From these two matrices of pairwise comparisons, we obtain two sets of weights. In a third matrix of comparisons, we attempt to say how much more important is research or teaching to the institute by obtaining relative weights or priorities for them. We then weight the vector of criteria relating to research with the priority of research, that relating to teaching with the priority of teaching. The result is the overall norm vector set by the committee for the candidates.

The next step is to compare candidates, if several have applied for the same position, with respect to each of the criteria under research and under teaching (one matrix for each criterion). We then weight the resulting vectors under the criteria by the priority of the criterion just derived. Finally, we add the weights obtained for each candidate to get the overall relative ranking with respect to the other candidates. Next we must evaluate whether the highest ranked individual measures up to the norms of the committee or does not qualify for the position.

To do this we use the full information gathered about the candidate as follows: we evaluate the candidate's standing with respect to the research criteria in a pairwise comparison matrix. The work may turn out to be more prolific than significant (although the committee in its standard evaluation may have emphasized significance over productivity). The net result is a vector of weights for the candidate corresponding to research work. Another vector is obtained for teaching ability. Again these two vectors may be rated by the committee's vector of priorities for research and teaching. Then the candidate's resulting vector is compared with the norm vector of the committee. The absolute deviation of corresponding components is calculated and each deviation is weighted by the priority of the corresponding criterion as set by the committee. As an indicator of the suitability of the candidate, the sum of these weighted deviations should be around (preferably less than) 10%. The following expression gives a formal representation of

Table 5

	Research	Theory	Principal eigenvector
Research	1	4	0.8
Teaching	1/4	1	0.2

the index measuring deviation from the norm:

$$d = \sum_{i=1}^n b_i |b_i - x_i|,$$

where b_i is the priority of the i th criterion, x_i is the

Table 6

	S	Q	R	P	N	V	G	Principal eigenvector
S	1	3	5	2	5	6	4	0.38
Q	1/3	1	2	2	4	5	4	0.21
R	1/5	1/2	1	2	3	3	3	0.14
P	1/2	1/2	1/2	1	3	3	3	0.13
N	1/5	1/4	1/3	1/3	1	2	2	0.06
V	1/6	1/5	1/3	1/3	1/2	1	1	0.04
G	1/4	1/4	1/3	1/3	1/2	1	1	0.04
Consistency index								0.06

S: Significance; P: Provocativeness; Q: Quality; N: Number; G: Growth; R: Reputation; V: Variety.

Table 7

	K	C	O	CO	E	I	CR	A	Principal eigenvector
K	1	1	3	2	3	6	4	5	0.25
C	1	1	2	2	2	2	5	4	0.2
O	1/3	1/2	1	1	4	4	5	3	0.16
CO	1/2	1/2	1	1	3	4	3	3	0.15
E	1/3	1/2	1/4	1/3	1	2	2	2	0.08
I	1/6	1/2	1/4	1/4	1/2	1	1/2	3	0.06
CR	1/4	1/5	1/5	1/3	1/2	2	1	4	0.06
A	1/5	1/4	1/3	1/3	1/2	1/3	1/4	1	0.04
Consistency index									0.10

K: Depth of knowledge; O: Openness to new ideas; CR: Ability to develop the critical ability of the students; C: Commitment to the subject; CO: Coherence; I: Influence on students; A: Accessibility to students; E: Ability to encourage weak students.

Table 8

Research							
Significance	Quality	Reputation	Provocativeness	Number	Variety	Growth	
0.31	0.17	0.11	0.10	0.05	0.03	0.03	
Teaching							
Knowledge	Commitment	Openness	Coherence	Encouragement	Influence	Critical Ability	Accessability
0.05	0.04	0.03	0.03	0.03	0.01	0.01	0.01

corresponding priority in the candidate's vector.

As an illustration, we derive the norm vector for professorship by a hypothetical committee and then compare it with the result for a candidate. The hypothetical committee has been charged by its dean to place preponderant emphasis on research. In that case, the pairwise comparison matrix of research vs. teaching as far as the University's quest for distinction is concerned (as perceived by its leadership) is given in Table 5.

Next we compare the seven research criteria mentioned above and obtain the comparison matrix of Table 6. Similarly the matrix corresponding to the teaching criteria is given by Table 7.

We now have two "ideal or standard" vectors, one corresponding to the research work and the other to the teaching ability for the position of a professor. As noted earlier, the respective priorities of the research work and teaching ability are 0.8 and 0.2. Thus, we weight the two vectors by these numbers and combine

them to get a vector with fifteen elements; seven corresponding to research work and eight to teaching ability. See Table 8. This composite vector is the norm for evaluating the suitability of all the prospective candidates.

In comparing this norm with a candidate's vector, we may find that although a candidate is not qualified to be offered the post, his or her composite vector of the distribution of abilities over the criteria may be a very close fit to the norm vector. Thus a candidate's vector should be compared with the norm only when the candidate has been found to be suitably qualified for consideration.

Our hypothetical winning candidate has the matrices of Table 9. The composite vector is given by Table 10. The index of deviation is: 0.01, which is very small when compared with the 10% level of acceptance. The candidate obviously deserves a professorial appointment if these are the only criteria to be considered.

Table 9

For research:

	S	Q	R	P	'N	V	G	Principal eigenvector
S	1	3	5	2	4	4	4	0.35
Q	1/3	1	3	2	5	4	3	0.22
R	1/5	1/3	1	3	2	3	2	0.14
P	1/2	1/3	1	4	1	2	0.11	0.11
N	1/4	1/5	1/2	1/4	1	1	1	0.05
V	1/4	1/4	1/3	1	1	1	1	0.06
G	1/4	1/3	1/2	1/2	1	1	1	0.06
							Consistency index	0.09

For teaching:

	K	C	O	CO	E	I	CR	A	Principal eigenvector
K	1	2	3	2	2	5	3	5	0.27
C	1/2	1	3	2	2	4	5	2	0.21
O	1/3	1/3	1	2	3	1	5	3	0.14
CO	1/2	1/2	1/2	1	2	3	2	3	0.12
E	1/2	1/2	1/3	1/2	1	1	2	1	0.07
I	1/5	1/4	1	1/3	1	1	2	1	0.07
CR	1/3	1/5	1/5	1/2	1/2	1/2	1	2	0.05
A	1/5	1/2	1/3	1/3	1	1	1/2	1	0.07
								Consistency index	0.09

Table 10

Research							
Significance	Quality	Reputation	Provocativeness		Number	Variety	Growth
0.3	0.18	0.11	0.09		0.04	0.05	0.05
Teaching							
Knowledge	Commitment	Openness	Coherence	Encouragement	Influence	Critical ability	Accessibility
0.04	0.04	0.03	0.02	0.01	0.01	0.01	0.01

6. Rationing energy to industries: optimization [15]

Only a short time ago it was unthinkable and deemed as an academic exercise to speak of rationing energy. It was felt that there could not be a crippling energy crisis because our energy czars and planners would presumably take our needs into their projections. Today, things look very different. Witness the cases of the lack of natural gas in the cold winter of 1976–1977 which caused the shut down of some schools and industries and the coal strike of 1977–1978.

In any case we must face the needs of our homes, offices, industries, and massive transportation systems not simply by making additional supplies of energy available but also by replanning and redesign to improve efficiency and to diminish the 7 percent annual rise in energy consumption.

Besides improving efficiency, for the long range we need to consider several alternatives to prevent severe energy shortages. Among them are:

1) Reduction of U.S. consumption to the level of domestic oil production.

2) Discovery or development of new forms of energy such as coal gasification, geothermal, nuclear fission, nuclear fusion and solar. But these forms are now in short supply.

3) Rationing. Although rationing is not an attractive alternative, we have seen that in cases of severe weather, energy had to be diverted from schools and industries in the Midwest to accommodate homeowner needs. Rationing can become a pressing alternative if supplies dry up or continuity in importing oil is seriously threatened.

In this application, we confined our analysis to manufacturing industries.

Examples of the groups we considered are: 1) food and kindred products, 2) tobacco manufacturers, 3) textile mill products, 4) apparel and related products, 5) lumber and wood products, 6) furniture and fixtures, 7) paper and allied products, 8) printing and publishing, 9) chemicals and allied products, 10) petroleum and coal products, etc.

The optimal weights generated for these classes of industries are applicable on a yearly basis and, therefore, the actual scheduling of allocation on a day-to-day or a month-to-month basis is not made explicit by the model. Our approach can be extended to peak power demand considerations where shortage of power may occur in a short time duration. In this case, the optimal scheduling of power and its allocation will be determined as a function of time.

We used the following objectives which fall into two classes: Class 1, characterized by two measurable indicators; contribution to economic growth (measured in dollars), and impact on the environment (measured in tons of pollutants). Class 2, characterized by three qualitative indicators; contribution to national security, to health, and to education. The measures for these were derived using judgments and the eigenvalue procedure.

The results of the two classes were composed hierarchically to obtain an overall priority for each industrial group.

As the real-life problem is too long to work out here we have chosen an example which illustrates how one does a rationing problem. It combines priorities and optimization.

The problem in the energy demand allocation is concerned with finding allocation weights for several large users of energy according to their overall contribution to different goals of society. Let us assume the following conditions:

There are three large users of energy in the United States: C_1 , C_2 , and C_3 . The goals against which these energy users will be evaluated are: Contribution to Economic Growth, Contribution to Environmental Quality and Contribution to National Security. Based on the overall objective of social and political advantage the matrix of paired comparisons of these three goals on the previously described scale from 1–9 is given by (we have forced consistency here):

	Econ	Env	Nat Sec
Econ	1	5	3
Env	1/5	1	3/5
Nat Sec	1/3	5/3	1

The normalized eigenvector corresponding to the dominant eigenvalue $\lambda = 3$ of this matrix is given by:

$$P(0) = \begin{bmatrix} 0.65217 \\ 0.13044 \\ 0.21739 \end{bmatrix}$$

The decision-maker, after a thorough study, has made the following assessment of the relative importance of each user from the standpoint of the economy, environment, and national security. The matrices giving these judgments are given in Table 11.

The corresponding normalized eigenvectors are respectively the three columns of the following matrix:

0.64833	0.09382	0.53962
0.22965	0.16659	0.29696
0.12202	0.73959	0.16342

This matrix is multiplied by the vector $P(0)$ yielding the following vector which is already normalized,

giving the eigenvector priorities of the activities C_1 , C_2 and C_3 :

$$\alpha \equiv \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0.55237 \\ 0.23606 \\ 0.21157 \end{bmatrix}.$$

We cannot allocate energy in proportion to the priorities of the industries as they may be interdependent. Material from a low priority industry may flow to a higher priority one. To express the relationship as a constraint we use the following input–output matrix:

	C_1	C_2	C_3
C_1	1.09730	0.22680	0.19020
C_2	0.07990	1.06570	0.06010
C_3	0.03950	0.33210	1.20710

When the coefficient in the (i, j) position of the above matrix is weighted by α_i and α_j and summed over each row, we obtain the vector of dependence numbers:

$$\beta \equiv \begin{bmatrix} 0.38659 \\ 0.07280 \\ 0.07523 \end{bmatrix}.$$

Suppose that the energy requirements R_i (in trillion B.T.U.) of the three users are as given in Table 12. Also assume that the total energy available has been cut back to a level of $R = 12,000$ B.T.U. We have the following linear programming problem:

Maximize

$$z = 0.38659 w_1 + 0.07280 w_2 + 0.07523 w_3,$$

whose coefficients are the corresponding elements of the vector β , subject to:

$$0 \leq w_1 \leq 0.38467,$$

$$0 \leq w_2 \leq 0.58575,$$

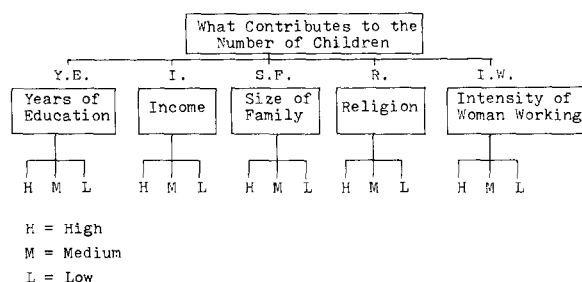
$$0 \leq w_3 \leq 0.27475,$$

in which the quantities on the right are respectively

Table 11

	Economy			Environment			National security		
	C_1	C_2	C_3	C_1	C_2	C_3	C_1	C_2	C_3
C_1	1	3	5	1	1/2	1/7	1	2	3
C_2	1/3	1	2	2	1	1/5	1/2	1	2
C_3	1/5	1/2	1	7	5	1	1/3	1/2	1

Activity (C_i)	Energy requirements (R_i)
C_1	4616
C_2	7029
C_3	3297
Total	14942

 $R_i/R, i = 1, 2, 3$ and to

$$w_1 + w_2 + w_3 = 1.$$

The optimal allocation is given by:

$$w_1 = 0.38467,$$

$$w_2 = 0.34058,$$

$$w_3 = 0.27475.$$

Thus only C_2 is not given its full requirement.

Note that here we have simplified the linear programming problem to make it easier to grasp the procedure.

7. Satisficing

The concept of optimizing as the best way of solving problems has been questioned in recent years and a substitute idea of satisficing has been proposed as a new way of approaching problems. In the following application we show how people's habits and proclivities lead them to outcomes that simply satisfy what they are pressing for, but not too much nor too little.

7.1. How many children in a family – 1945–1970

In a two hour exercise involving about a dozen people the majority of whom were graduate students, it was desired to determine the average number of children that an American family was likely to have in 1945–1970. The same approach may be used to compute the number of children for a specific family for then instead of making the variety of comparisons for all types of families one simply does it for one family. The hierarchy has the following structure with the pairwise comparison matrices, eigenvectors and consistency indices as indicated. High, medium and low were used as subcriteria and properly regarded for positive or negative contribution towards a large family. The five highest priority factors which contribute to having a large family were selected as criteria for comparing the number of children that a family is likely to have, i.e., 1, 2, 3, 4, 5 or more. Of course if all the factors (not just the highest priority ones) were included in the last comparison, this number would be reduced. This is an illustrative example and not a conclusive study. See Fig. 1 and Tables 13–18. Com-

How many children	Y.E.	I.	S.F.	R.	I.W.	Eigenvector
Y.E.	1	1	5	4	7	0.38
I.	1	1	5	4	7	0.38
S.F.	1/5	1/5	1	1/3	1/3	0.05
R.	1/4	1/4	3	1	2	0.12
I.W.	1/7	1/7	3	1/2	1	0.07
					$\lambda_{\max}$	5.257
					Consistency index	0.06

Table 14

Y.E.	H	M	L	Eigen-vector
H	1	1/3	1/6	0.095
M	3	1	1/3	0.25
L	6	3	1	0.65
			$\lambda_{\max}$	3.018
			Consistency index	0.01

Table 15

I.	H	M	L	Eigen-vector
H	1	3	1/3	0.26
M	1/3	1	1/5	0.10
L	3	5	1	0.64
			$\lambda_{\max}$	3.039
			Consistency index	0.02

Table 16

S.F.	H	M	L	Eigen-vector
H	1	3	5	0.64
M	1/3	1	3	0.26
L	1/5	1/3	1	0.10
			$\lambda_{\max}$	3.039
			Consistency index	0.02

Table 17

R	H	M	L	Eigen-vector
H	1	4	5	0.68
M	1/4	1	2	0.20
L	1/5	1/2	1	0.12
			$\lambda_{\max}$	3.025
			Consistency index	0.01

Table 18

I.W.	H	M	L	Eigen-vector
H	1	3	8	0.67
M	1/3	1	4	0.26
L	1/8	1/4	1	0.07
			$\lambda_{\max}$	3.018
			Consistency index	0.01

posing weights in the usual manner for the hierarchy we obtain the following matrix.

	Y.E.	I.	S.F.	R.	I.W.
H	0.036	0.099	0.032	0.082	0.047
M	0.095	0.038	0.013	0.024	0.018
L	0.247	0.243	0.005	0.014	0.005

By examining this table we note that the highest priority factors which contribute to large family size are essentially five. They and their priorities are given in Table 19.

We now construct the pairwise comparison matrices of the number of children likely to result from each of these factors. (Note: L-Y.E. and L-I. have the same comparison matrices.) See Table 20.

Since we are interested in estimating the expected number of children per family in the United States, we must examine the demographic distribution of the five factors we are considering. We have the matrix of Table 21.

We now use this eigenvector to smooth (weight) the above normalized eigenvector for the five criteria. We have Table 22.

Finally we multiply the matrix of eigenvectors for family size by the normalized vector yielding Table 24. The expected number of children is given by

$$0.087x^1 + 0.191x^2 + 0.282x^3 + 0.292x^4 + 0.150x^5 = 3.23.$$

The actual statistics for the average number of children born to women depends on the year in which the woman was born. We have Table 25. The last figure in the table is taken from ref. [17]. The remaining figures are from ref. [18].

Considering that most of the students were born to women born in the period 1923–1932, the result obtained through the hierarchy is very close and indicates that the children are probably influenced in looking back by the attitudes of their family.

7.2. The number of children in an average family – 1970–

This exercise was carried out with the participation of four highly knowledgeable and well informed

Table 19

Factors	L–Y.E.	L–I.	H–I.	M–Y.E.	H–R.
Priorities	0.247	0.243	0.099	0.095	0.082
Normalized priorities	0.323	0.318	0.129	0.124	0.107

Table 20

L–Y.E. L–I.	1	2	3	4	5	Eigen- vector	H–I.	1	2	3	4	5	Eigen- vector
1	1					0.04	1	1					0.08
2	4	1				0.10	2	4	1				0.25
3	6	3	1			0.24	3	5	3	1			0.45
4	8	5	3	1		0.41	4	4	1/2	1/3	1		0.18
5	6	3	1	1/3	1	0.21	5	1/4	1/6	1/7	1/5	1	0.04
M–Y.E.	1	2	3	4	5	Eigen- vector	H–R	1	2	3	4	5	Eigen- vector
1	1					0.17	1	1					0.04
2	3	1				0.35	2	4	1				0.09
3	3	1	1			0.35	3	5	3	1			0.20
4	1/3	1/4	1/4	1		0.08	4	7	5	3	1		0.47
5	1/5	1/6	1/6	1/2	1	0.05	5	5	3	1	1/3	1	0.20

Table 21

Distribution in the U.S.	L–Y.E.	L–I	H–I	M–Y.E.	H–R	Eigenvector
L–Y.E.	1					0.175
L–I	1	1				0.175
H–I	1/7	1/7	1			0.032
M–Y.E.	5	5	7	1		0.547
H–R	1/4	1/4	5	1/7	1	0.072
					$\lambda_{\max}$	5.52
					Consistency index	0.13

Table 23

	L–Y.E.	L–I	H–I	M–Y.E.	H–R
Demographically eigenvector	0.057	0.056	0.004	0.068	0.008
Normalized vector	0.295	0.290	0.021	0.352	0.042

Table 24

Number of children				
1	2	3	4	≥5
0.087	0.191	0.282	0.292	0.150

couples about the prospects of the number of children in a family looking ahead from the 1970's. The criteria and their pairwise comparison matrix are given in Table 26.

We normalize the eigenvector after eliminating the fifth factor whose weight is negligible. We have Table 27. We then compare the number of possible children in a family as influenced by each criterion assuming that no family is likely to have more than four children (or that those which do are relatively small in number): Table 28. The final weighted vector is given by Table 29. The expected number of children for a prospective family is:

$$0.028 x^0 + 0.174 x^1 + 0.495 x^2 + 0.239 x^3 + 0.064 x^4 = 2.14$$

This is a reasonable result which corresponds to prevailing trends (projecting 2.11 children). Note that our criteria in this case are not influenced by demographic distribution.

8. Higher education in the United States (1985–2000): the forward planning process

This description [11] is based on an experiment conducted by twenty-eight college level teachers,

Table 26

Contribution to family size	Av.	W.M.	O.M.	E.M.	C.R.	S.P.	Eigenvector
Av.	1	7	5	4	8	4	0.481
W.M.	1/7	1	1/2	1/2	4	1/2	0.074
O.M.	1/5	2	1	1/2	6	1/2	0.108
E.M.	1/4	2	2	1	6	1	0.154
C.R.	1/8	1/4	1/6	1/6	1	1/6	0.028
S.P.	1/4	2	2	1	6	1	0.154
$\lambda_{\max}$							6.247
Consistency index							0.05

Av. = Availability of contraceptives; W.M. = Working mother; O.M. = Older age of motherhood; E.M. = Education of mother; C.R. = Cost of raising children; S.P. = Social Pressure.

Table 25

Year woman is born	Average number of children
1898–1902	2.53
1903–1907	2.34
1908–1912	2.29
1913–1917	2.45
1918–1922	2.86
1923–1927	3.10
1928–1932	3.14

mostly from the mathematical sciences under the leadership of the author. The problem was to examine the future of higher education in the U.S. to see whether the participants' job security would be threatened.

Fig. 2 presents the hierarchical structure of the factors, actors and their motivating objectives which the group saw as chain of influences which would affect the form that higher education will take between 1985 and 2000. No strict definitions of the various terms will be given although during the development (which took approximately 9 hours) comments were made on some of the intended meanings.

Seven scenarios are offered:

- (1) (PROJ) 1985-Projection of the present status quo (slight perturbation of present)
- (2) (VOTEC) Vocational-Technical Oriented (skill orientation)
- (3) (ALL) Education for All (subsidized education)
- (4) (ELITE) Elitism (for those with money or exceptional talent)
- (5) (APUB) All Public (government owned)

Table 27

Av.	W.M.	O.M.	E.M.	S.P.
0.495	0.077	0.112	0.159	0.159

- (6) (TECH) Technology Based (little use of class-room – use of media, computers)
 (7) (P.T.) Part-Time Teaching – no research orientation.

The characteristics which were considered and which were calibrated so as to give profiles of the various scenarios are given in Table 30 which is presented here for the sake of brevity includes all the information we need to find out about the prospects

Table 29

Number of children	0	1	2	3	4
Priority	0.028	0.174	0.495	0.239	0.064

of job security for the faculty. The calibration numbers are integers between –5 and 5. These measurements were arrived at by consensus.

Zero (0) represents things as they now are (in the group's opinion). Positive integers represent the various degrees of "increasingness" or "more than now". Negative integers represent various degrees of "decreasingness" or "less than now". For example under Institution-Governance Structure we see a 5 for Scenario 6.

Table 28

Av.	0	1	2	3	4	Eigen-vector	W.M.	0	1	2	3	4	Eigen-vector
0	1					0.026	0	1					0.022
1	8	1	Reciprocal			0.126	1	9	1	Reciprocal			0.154
2	8	6	1			0.526	2	9	4	1			0.393
3	9	3	1/4	1		0.250	3	9	6	1/2	1		0.362
4	7	1/3	1/7	1/6	1	0.072	4	9	1/6	1/6	1/7	1	0.068
						$\lambda_{\max} = 5.686$							$\lambda_{\max} = 5.965$
						Consistency index = 0.17							Consistency index = 0.24
O.M.	0	1	2	3	4	Eigen-vector	E.M.	0	1	2	3	4	Eigen-vector
0	1					0.053	0	1					0.022
1	8	1	Reciprocal			0.261	1	9	1	Reciprocal			0.151
2	8	5	1			0.542	2	9	7	1			0.501
3	6	1/5	1/7	1		0.122	3	9	4	1/3	1		0.264
4	1/7	1/9	1/9	1/8	1	0.022	4	9	1/7	1/8	1/6	1	0.063
						$\lambda_{\max} = 6.13$							$\lambda_{\max} = 6.089$
						Consistency index = 0.28							Consistency index = 0.27
						S.P.	0	1	2	3	4	Eigen-vector	
						0	1					0.031	
						1	9	1	Reciprocal			0.295	
						2	7	1	1			0.408	
						3	7	1	1/4	1		0.201	
						4	4	1/6	1/7	1/4	1	0.065	
												$\lambda_{\max} = 5.349$	
												Consistency index = 0.09	

LEVELS

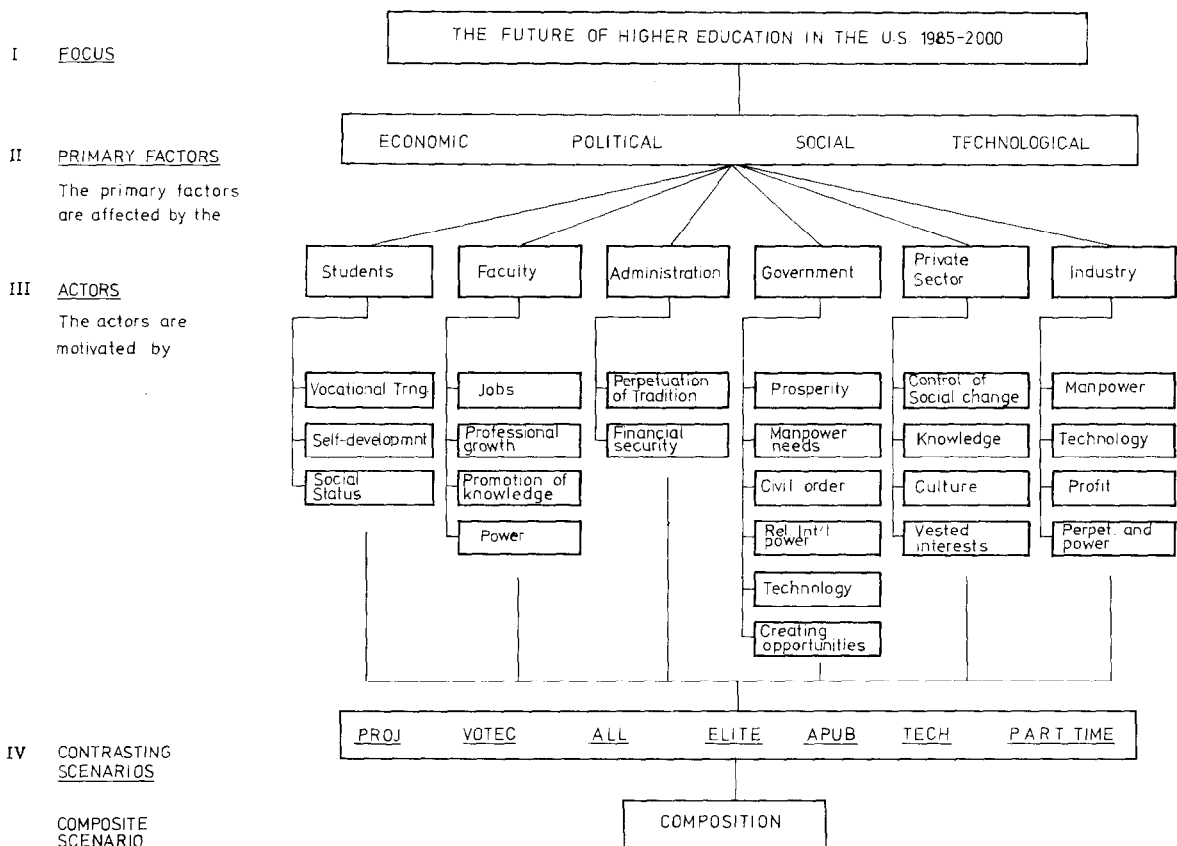


Fig. 2. A hierarchy of influences on higher education.

This means that the group thought that there would be a very large measure of administrative control (relative to the state of things at present) in a technology based higher education system in 1985 and after. On the other hand, if Scenario 2 (Education—Value of a Degree) would diminish considerably (−2) compared to how it is valued today. The weights of the scenarios and the column composite weight need to be determined. That is the purpose of the exercise. Note that the answers are already given at the very top row of Table 30 and the last column under COMP.

For the pairwise comparison matrix of the primary factors we answer the following question by assigning numerical judgments: Which factor has the greater impact on higher education? The resulting eigenvector

was:

$$(\text{Econ., Pol., Soc., Tech.}) = (0.549, 0.106, 0.236, 0.109)$$

Thus the group considered the economic factor somewhat dominant over the other three factors.

The next set of questions followed in the comparison of the actors with respect to each primary factor were:

Who (what) has more impact on the way education affects the economy of the United States?

Who (what) has more impact on the way education affects the political situation in the United States?

Who (what) has more impact on the way education affects the social issues in the United States?

Who (what) has more impact on the way education

Table 30
Seven scenarios and the calibration of their characteristics. Scale: -5 to +5.

Scenario weights	0.096	0.259	0.191	0.174	0.122	0.068	0.081	COMP
Characteristics	1 PROJ	2 VOTEC	3 ALL	4 ELITE	5 APUB	6 TECH	7 P.T.	
Students								
1. Number	-2	+2	+4	-3	-1	+2	-2	0.42
2. Type (I.Q.)	-1	-2	-3	+3	-1	-2	-1	-1.0
3. Function	+1	-1	0	+1	0	-2	+2	0.03
4. Jobs	+1	+4	-3	+4	+1	-2	+1	1.32
Faculty								
1. Number	-2	+2	+4	-3	-1	-5	-4	-0.22
2. Type (Ph.D.)	+1	0	-2	+3	+1	+2	-3	0.25
3. Function (role on campus)	-2	-3	-2	+1	-2	-5	-5	-2.12
4. Job security	-2	+1	+2	-3	-1	-4	-4	-0.79
5. Acad. Freedom	0	-2	0	+2	-1	-4	-5	-0.97
Institution								
1. Number	-1	+2	+2	-3	-1	-4	-1	-0.19
2. Type (acad/non-acad)	-1	-4	-3	+3	-1	-3	-3	-1.75
3. Governance	+2	+4	+1	-2	+2	5	5	2.06
4. Efficiency	+2	+3	-2	+4	-1	-1	0	1.09
5. Accessibility	0	+2	+5	-3	+2	+4	+1	1.55
6. Culture-entertain	0	-2	+3	+3	+1	-3	-1	0.41
7. Avail \$ and other resources	-1	+2	+2	-2	0	-1	-3	0.64
Education								
1. Curriculum (lifelong learning)	1	-2	+2	+3	+1	+0	-1	0.50
2. Length of study	0	-3	+2	0	+1	+2	0	-0.14
3. Value of a degree	-1	0	-2	+4	-1	-2	-2	-0.20
4. Cost per student	+3	+3	+4	+2	-1	-1	-1	-2.43
5. Research by faculty	+1	-1	-1	+3	+1	-3	-4	0.24

affects the technology of the United States?

The matrix of eigenvectors and the composite weight of the actors with respect to their impact on

the future of higher education are given in Table 31.

Since government and industry were by far the dominant actors we focus only on these two weighting

Table 31

Econ.	Pol.	Soc.	Tech.					
S	0.04	0.04	0.10	0.02	0.55	E	0.05	S
F	0.02	0.04	0.07	0.10			0.04	F
A	0.06	0.03	0.04	0.03	0.11	P =	0.05	A
G	0.47	0.49	0.41	0.23	0.24	S	0.44	G
P	0.12	0.12	0.16	0.16			0.13	P
I	0.28	0.29	0.26	0.44	0.21	T	0.30	I

Table 32

For government:	0.44	0.20	0.09	Prosperity
		0.52	0.23	Civ. order
		0.09	= 0.02	Manpower
		0.11	= 0.05	R.I.P.
		0.05	0.02	Technology
For industry:	0.30	0.03	0.01	Crezle oppor.
		0.04	0.01	Manpower
		0.08	= 0.02	Technology
		0.33	= 0.10	Profit
		0.55	0.17	Perpet. and power

the eigenvector of their objectives (obtained from separate pairwise comparison matrices) by their weights of 0.44 and 0.30 respectively as in Table 32.

From this we see that the most influential objectives are prosperity and civil order for Government and profit and perpetuation and power for Industry. Using these four objectives and normalizing their weights we get the weight vector:

0.15	Prosperity
0.39	Civil Order
0.17	Profit
0.29	Perpetuation and Power

This vector will be used to get our scenario weights.

The final step necessary to get our scenario weights is to construct the dominance matrices for the seven scenarios with respect to each of the objectives (four in our case).

The questions asked for the matrices of the scenarios with respect to these four objectives in order to construct pairwise comparison matrices were respectively:

Which scenario has more impact on the prosperity of the United States?

Which scenario has more impact on the civil order of the United States?

Which scenario has more impact on profitability?

Which scenario has more impact on perpetuating industrial methods and power?

The eigenvectors of the impact of the scenarios on the objectives and the composite weights of the scenarios are as in Table 33.

We now obtain the composite scenario: a single scenario obtained by finding composite scale measurement for each of the characteristics. The composite scale measurement for a characteristic is obtained by forming the sum of the products of scenario weight by the corresponding characteristic measurement. For example, for the number of students we have:

$$(-2)(0.096) + (2)(0.259) + (4)(0.191) + (-3)(0.174)$$

$$(-1)(0.122) + (2)(0.068) + (-2)(0.081) = 0.42$$

This measurement is found in the table in the last column on the right. Similarly for the other characteristics we shall not go into the detail of interpreting the results. Suffice it to say that faculty job security will be only very slightly threatened due to tight governance and tighter resources. The group was satisfied with the outcome despite prolonged debate and considerable bargaining and compromise.

9. An observation relating space, time and hierarchies

Let us note the following regarding the analytical output of hierarchical measurement.

The composition of weights in a hierarchy yields

Table 33

		PROS	C. ORD.	PROF.	P and P			SC.
Scenario	1	0.129	0.125	0.067	0.062	=	0.096	1
	2	0.329	0.180	0.309	0.306		0.259	2
	3	0.275	0.369	0.028	0.026		0.191	3
	4	0.041	0.033	0.331	0.330		0.174	4
	5	0.149	0.177	0.048	0.085		0.122	5
	6	0.032	0.050	0.129	0.075		0.068	6
	7	0.045	0.065	0.089	0.115		0.081	7

multi-linear expressions (a covariant tensor) of the form:

$$w_i^h = \sum_{i_2, \dots, i_{h-1}=1}^{N_{h-1}, \dots, N_1} w_{i_1 i_2}^h w_{i_2 i_3}^{h-1} \dots w_{i_{h-2} i_{h-1}}^2 w_{i_{h-1}}^1, i_1 \equiv i$$

for the priority of the i th element in the h th level of the hierarchy. The composite vector W^h for the entire h th level is represented by the covariant hypertensor (a vector with tensor components). Similarly the left eigenvector approach to a hierarchy gives rise to a contravariant hypertensor.

The classical problem of relating space (geometry) and time to subjective thought (see ref. [19]) can perhaps be examined (we are working on it) by showing that the functions of mathematical analysis (and hence also the laws of physics) are derivable as truncated series from the above tensors by composition in an appropriate hierarchy. We recall that in dimensional analysis there is a theorem which asserts that all physical quantities may be expressed as products of powers of primary variables.

10. Conclusion

In the first paper [16] we discussed the forward and backward processes of planning. Here we illustrated each of the processes by a separate example but did not bring the two together. The interaction of forward and backward processes requires that one actor identifies his desired futures and works backwards to find policies which he sees can bring about such a future, determines which of the other actors is a most promising source of influence and attempts to bring about changes in that actor's policies. He then works forward again to see what mix of outcome results from the change and compares it with his desired outcome. He now either modifies his desired outcome or looks for more effective policies and other actors to influence attempting to produce convergence of the two outcomes to some kind of compromise. Of course this is a dynamic process whose outcomes tend to change with the implementation of new policies.

To summarize we have found people to be enthusiastic in structuring a problem and providing the judgments for its outcome. The approach seems to have a wide variety of potential applications and is a

convenient vehicle for bringing together the theory and practice of modeling.

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